A group representation on a Hilbert space of analytic functions

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MACKEY [2] HAS DEFINED the Laplace Transform for Locally Compact Abelian groups. In particular he obtains a correspondence between functions defined on the group which are square integrable with respect to an exponential weight function and functions analytic in a sense analogous to the usual definition. In this paper we use that correspondence to construct a representation of the group on a Hilbert space of the latter functions. We also have that the analyticity does imply a number of properties akin to analyticity for functions of several complex-variables.

NOTATION AND TERMINOLOGY

Let G be a Locally Compact Abelian group and $\hat{G}$ its character group. It is well-known that $\hat{G}$ is also Locally Compact Abelian since G is. We will write both G and $\hat{G}$ in multiplicative form. Denote by $\bar{G}$, the real valued, continuous linear functionals on G, i.e.

$$\bar{G} = \{x | x(g) \quad \text{real for} \quad g \in G, \quad x \text{ is continuous} \\ x(gh) = x(g) + x(h) \quad g, h \in G\}.$$

A subset $K \subset \bar{G}$ is said to be large if $0 \in K$ and the closed linear span of K is $\bar{G}$. For example, if $f \in L^2(G)$ with the usual Haar measure and K_f is the

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set of $x \in \bar{G}$ such that $\exp(-x(g))f(g) \in L^2(G)$ then K_f is a convex subset of $\bar{G}$. Further if G happens to be the reals (under addition) then $\hat{G}$ is isomorphic with the reals and so is $\bar{G}$, K_f will be an interval containing 0 and also K_f will be large. By a theorem of Hamburger [4], pp. 245 this interval corresponds to the strip of analyticity of the Laplace Transform of f. We shall call x an interior point of K if there exists a real number $u > 1$ such that $ux \in K$.

FUNCTIONS ON G

In this section we will only be interested in functions contained in $L^2(G)$ but we will assume that G is fixed and it will be necessary to consider different sets $K \subset \bar{G}$ so that we abuse the notation slightly as follows. Let K be a large convex subset of $\bar{G}$ such that each point of K is an interior point then we write

$$L^2(K) = \{f|, K_f \supset K\} \quad \text{where} \quad K_f = \{x|\exp(-x(g))f(g) \in L^2(G)\}.$$

Mackey has used the term "strongly in L^2" to denote the functions for which K_f is large convex, it follows that all of the members of $L^2(K)$ are "strongly in L^2". With the semi-norms

$$\|f\|_x^2 = \int_G |f(g)\exp(-x(g))|^2\, dg.$$

$L^2(K)$ becomes a complete linear topological space and convolution is a jointly continuous multiplication.

THE HILBERT SPACE H²(K)

$\bar{G} \times \bar{G}$ becomes a complex linear space if multiplication by a scalar is defined by $(u + iv)(x_1, x_2) = (ux_1 - vx_2, ux_2 + vx_1)$. With this multiplication, $\bar{G} \times \bar{G}$ is like the complex plane or more accurately, like finite or infinite dimensional complex space. However it is not convenient to consider functions defined on $\bar{G} \times \bar{G}$ but rather on $K \times \hat{G}$ since for $f \in L^2(K)$, $(x, y) \in K \times \hat{G}$ the Laplace Transform of f is

$$F(x, y) = \int_G \exp(-x(g))\,\overline{(y, g)}\,f(g)\, dg.$$

$\bar{G}$ is used however to define analyticity using a Frechet-type derivative.

DEFINITION 1 Let K be a large convex subset of $\bar{G}$ and x an interior point of K, $F(x, y)$ a complex-valued function defined on $K \times \hat{G}$. F is said to be analytic at x if

(i) $$F_{(x_1, x_1)}(x, y) = \operatorname*{Lim}_{u \to 0} \frac{F(x + ux_1, yx_2[u]) - F(x, y)}{u} \quad \text{exists for all}$$

$$(x_1, x_2) \in \bar{G} \times \bar{G}$$

(ii) $F_{(x_1, x_2)}$ is a complex-homogeneous function of (x_1, x_2).

$$(x_2[u])(g) = \exp(iux_2(g))$$

and hence is a one parameter subgroup of G [2].

As has been shown in [3], (ii) corresponds to the Cauchy–Riemann equations. For K large convex such that each point is an interior point denote

$$H^2(K) = \left\{F \,|\, F \text{ analytic in } K \times \hat{G}, \sup_{x \in K} \int_{\hat{G}} |F(x, y)|^2 \, dy < \infty\right\}$$

CORRESPONDENCE THEOREM, [2]

$F \in H^2(K)$ implies there exists $f \in L^2(K)$ such that (modulo functions zero almost everywhere)

$$F(x, y) = \int_G \exp(-x(g)) \overline{(y, g)} f(g) \, dg.$$

Actually, Mackey's Theorem is stronger, if instead of $H^2(K)$ we take the class of functions analytic on $K \times \hat{G}$ and in $L^2(G)$ for every $x \in K$ then there is a 1 : 1 correspondence between these functions and the set $L^2(K)$.

In [4], this author has shown that $H^2(K)$ is a Hilbert space and that the elements are truly analytic in the sense of being infinitely differentiable including mixed derivatives. The proofs depend on showing that the norm topology is stronger than compact-open topology and constructing related functions of a complex variable which are analytic if and only if the original function was, in the sense of Definition 1.

THE GROUP REPRESENTATION

For $h \in G$ define T_h acting on $H^2(K)$ as follows

$$T_h F(x, y) = \int_G \exp(-x(g)) \overline{(y, gh)} f(g) \, dg$$

where f is the function in $L^2(K)$ given by the Correspondence Theorem. From the linearity of the integral it follows easily that

$$T_h(\alpha F_1 + \beta F_2) = \alpha(T_h F_1) + \beta(T_h F_2).$$

Since G is an abelian group we have

$$\begin{aligned} T_{h_1}(T_{h_2}F) &= T_{h_2}(T_{h_1}F) \\ &= T_{h_1 \cdot h_2}F \end{aligned}$$

so that $T_e = I$ and $(T_h)^{-1} = T_{h^{-1}}$. The inner product determined by the norm of $H^2(K)$ is

$$(F_1, F_2) = \sup_{x \in K} \int_{\hat{G}} F_1(x, y)\, \overline{F_2(x, y)}\, dy$$

and therefore

$$(F_1, T_{h^{-1}}F_2) = \sup_{x \in K} \int_{\hat{G}} F_1(x, y)\, \overline{T_{h^{-1}}F_2(x, y)}\, dy$$

$$= \sup_{x \in K} \int_{\hat{G}} \left[\int_G \exp(-x(g))\, \overline{(y, g)}\, f_1(g)\, dg\right] \overline{\left[\int_G \exp(-x(t)\, (y, th)\, f_2(t)\, dt\right]}\, dy$$

but $\overline{(y, th)} = \overline{(y, t)}\,\overline{(y, h)}$ so that $(F_1, T_{h^{-1}}F_2) = (T_h F_1\ F_2)$ and hence

$$T_{h^{-1}} = (T_h)^{-1} = (T_h)^*$$

so the representation is unitary. Also we find $||T_h F|| = ||F||$ by straight forward computation.

Although it is not possible in general to define polynomials on $K \times \hat{G}$, $H^2(K)$ is non-empty since the characteristic functions of compact subsets of G have Laplace transforms which are in $H^2(K)$. Questions such as the existence of sufficiently many elements of $\bar{G}$ to separate points in G have been answered in [1] and [2].

References

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